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LETTER TO THE EDITOR

Quantum Euler–Manakov top on the 3-sphere S_3

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Abstract. We construct the harmonic polynomials on the real S_3-sphere associated with the separation of the Laplace operator in general ellipsoid coordinates. They describe a basis of an irreducible unitary representation of the o(4) Lie algebra coming from the quantum integrable system: the Euler–Manakov top on the 3-sphere S_3 or, equivalently, a 2-site su(2) *XYZ* Gaudin magnet. An isomorphism of this model with the 4-site su(1, 1) *XXX* Gaudin magnet is established. The final two-parameter spectral problem is efficiently solved.

Let us consider the quantum integrable system on the 3-sphere S_3 which is defined as a special case of the Euler–Manakov top on the o(4) Lie algebra, having a complete set of *quadratic* integrals of motion. There are two such tops on the S_3-sphere: the Manakov and Steklov ones (see, for example, [1]). Here we study the first one.

The quantum integrable system considered generates a non-subgroup-type basis of an irreducible unitary representation of the o(4) algebra, and is of interest in atomic physics [2].

Two commuting quadratic functions on the o(4) generators (a and b being arbitrary real constants)

$$X = s_1 t_1 + \frac{a-b-1}{1-a-b} s_2 t_2 + \frac{b-a-1}{1-a-b} s_3 t_3$$

$$Y = b(1-a)(s_2^2+t_2^2) + a(1-b)(s_3^2+t_3^2) + 2b(1-a)\frac{b-a-1}{1-a-b} s_2 t_2 + 2a(1-b)\frac{a-b-1}{1-a-b} s_3 t_3 \tag{1}$$

are considered as two integrals of motion and fix the system [2]. Here the generators s_i, t_i, $i=1,2,3$, obey the standard commutation relations ($\mathrm{o}(4) \simeq \mathrm{su}_s(2) \oplus \mathrm{su}_t(2)$):

$$[s_i, s_j] = \varepsilon_{ijk} s_k \qquad [t_i, t_j] = \varepsilon_{ijk} t_k \qquad [s_i, t_j] = 0. \tag{2}$$

The o(4) Casimir operators

$$C = -2(\bar{s}^2 + \bar{t}^2) \qquad \tilde{C} = 2(-\bar{s}^2 + \bar{t}^2) \tag{3}$$

supplement two integrals (1) to the complete set of mutually commuting operators.

Let us consider the following 2×2 L-operator:

$$L(u) = \sum_{i=1}^{3} [s_i \omega_i(u-\varkappa) + t_i \omega_i(u+\varkappa)]\sigma_i/2 \tag{4}$$

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where σ_i are the standard Pauli matrices, $\varkappa$ is a constant and the elliptic functions ω_i are expressed as follows:

$$\omega_1(u) = \frac{1}{\operatorname{sn}(u, k)} \qquad \omega_2(u) = \frac{\operatorname{dn}(u, k)}{\sin(u, k)} \qquad \omega_3(u) = \frac{\operatorname{cn}(u, k)}{\operatorname{sn}(u, k)}. \tag{5}$$

Let us introduce the following notation for the tensor product

$$\overset{1}{L}(u) = L(u) \otimes I \qquad \overset{2}{L}(v) = I \otimes L(v) \tag{6}$$

where I denotes a 2×2 unity matrix. Then the L-operator (4) satisfies the equation

$$[\overset{1}{L}(u), \overset{2}{L}(v)] = [r(u-v), \overset{1}{L}(u) + \overset{2}{L}(v)] \tag{7}$$

with the r-matrix having the form

$$r(u) = \frac{1}{2} \sum_{i=1}^{3} \omega_i(u, k) \sigma_i \otimes \sigma_i. \tag{8}$$

We can call the integrable system generated by the determinant of the L-operator (4) a 2-site su(2) XYZ Gaudin magnet [3].

The integrals of motion are extracted from an L-operator by means of a standard procedure, so we have

$$\begin{aligned} H_1 &= \sum_{i=1}^{3} \omega_i s_i t_i \\ H_2 &= \frac{1}{2} \sum_{i=1}^{3} \left(-\omega_i^2 (s_i^2 + t_i^2) + 2 \frac{\omega_1 \omega_2 \omega_3}{\omega_i} s_i t_i \right) \\ \omega_i &= \omega_i(2\varkappa). \end{aligned} \tag{9}$$

It can be easily shown that two integrals (9) are equivalent to the pair of previous ones (1) with the following correspondence:

$$\begin{aligned} X &= H_1/\omega_1 \\ Y &= -\frac{1}{(\omega_2+\omega_3)^2} \left(H_2 - \frac{2\omega_2\omega_3}{\omega_1} H_1 + \frac{\omega_1^2}{2} (\bar{s}^2 + \bar{t}^2) \right) \\ a &= \frac{\omega_1 + \omega_3}{\omega_2 + \omega_3} \qquad b = \frac{\omega_1 + \omega_2}{\omega_2 + \omega_3}. \end{aligned} \tag{10}$$

The equations (4)-(10) have the same form both in classical and quantum mechanics. The equations of motion in these two approaches are also identical and can be represented in the Lax form

$$\dot{L}(u) = [L(u), A(u)] \tag{11}$$

where the explicit form of A-matrix depends on the Hamiltonian chosen and is derived from the fundamental relation (7) [1].

We are looking for the common eigenfunctions of the operators X and Y. Consider the Hilbert space of square-integrable functions $f(\bar{x})$ on the 3-sphere S_3: $\Sigma_{\alpha=1}^{4} x_\alpha^2 = 1$. There the generators s_i, t_i can be written in terms of the differential operators $D_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha$, where ∂_α denotes $\partial/\partial x_\alpha$. The Hilbert space is the direct sum $\mathcal{H} \simeq \Sigma_{2s=0}^{\infty} \oplus \mathcal{H}_s$, where the $(2s+1)^2$-dimensional spaces $\mathcal{H}_s$ consist of the C^∞-functions

$$F(\bar{x}): \qquad \bar{s}^2 F = \bar{t}^2 F = -s(s+1) F \qquad (\bar{s}^2 \equiv \bar{t}^2 \text{ on } \mathcal{H}).$$

The problem will be to find the basis in $\mathscr{H}_s$. The Casimir operator $\tilde{C}$ is equal to zero, while C takes the form

$$C=-2(\bar{s}^2+\bar{t}^2)=-\Delta+P^2+2P \tag{12}$$

where $\Delta=\Sigma\,\partial_\alpha^2$ and $P=\Sigma\,x_\alpha\partial_\alpha$.

Let us introduce the system of ellipsoid coordinates ρ_i, $i=1,2,3$, on the 3-sphere S_3 [2] (e_α being arbitrary real constants)

$$\sum_{\alpha=1}^{4}\frac{x_\alpha^2}{\rho-e_\alpha}\stackrel{\text{def}}{=}0 \qquad x_\alpha^2=\frac{\Pi_{i=1}^3(\rho_i-e_\alpha)}{\Pi_{\beta\neq\alpha}(e_\beta-e_\alpha)}. \tag{13}$$

Notice that we choose e_α to be arbitrary, although one must put

$$e_1=0 \qquad e_2=1 \qquad e_3=a \qquad e_4=b \tag{14}$$

to deal with the operators X and Y (1). The ρ_i, e_α satisfy the inequalities

$$e_1<\rho_1<e_2<\rho_2<e_3<\rho_3<e_4.$$

This system is the most general orthogonal coordinate one on the 3-sphere S_3.

We consider the following spectral problem

$$\tfrac{1}{4}CF(\bar{x})=s(s+1)F(\bar{x}). \tag{15}$$

Changing the variables $x_\alpha\to\rho_i$, one obtains the factorization of the eigenfunction $F(\bar{x})$:

$$\tfrac{1}{4}C\prod_{i=1}^{3}f_i(\rho_i)=s(s+1)\prod_{i=1}^{3}f_i(\rho_i). \tag{15$'$}$$

The separation equations are [2]

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho_i^2}+\frac{1}{2}\sum_{\alpha=1}^{4}\frac{1}{\rho_i-e_\alpha}\frac{\mathrm{d}}{\mathrm{d}\rho_i}+\frac{-s(s+1)\rho_i^2+\lambda\rho_i+\mu}{\Pi_\alpha(\rho_i-e_\alpha)}\right)f_i(\rho_i)=0. \tag{16}$$

They have just the same form for the different i. The generalized Lamé differential equation (16) is of the Fuchsian type with four elementary regular singularities at e_α and a regular singularity at infinity with the exponents s and $-s-1$. It appears that the separation constants λ and μ in (16) are the eigenvalues of X and Y

$$XF(\bar{x})=\lambda F(\bar{x}) \qquad YF(\bar{x})=\mu F(\bar{x}) \tag{17}$$

with the set of e_α from (14). To construct F (and f_i) we will apply the following:

Theorem 1. A complete basis in $\mathscr{H}_s$ of $(2s+1)^2$ eigenfunctions $F(\bar{x})$ diagonalizing the operators C, X and Y is determined by

$$F(\bar{x})=\left(\prod_{j=1}^{K}x_{\alpha_j}\right)\prod_{q=1}^{M}\left(\sum_\alpha\frac{x_\alpha^2}{u_q-e_\alpha}\right) \tag{18}$$

where M and $K\in\mathbb{Z}_+=\{0,1,2,\ldots\}$, $K\leqslant 4$, and M and K are constrained by $2s=2M+K$; $\alpha_i\in\{1,2,3,4\}$, $\alpha_i\neq\alpha_j$ if $i\neq j$. The real parameters u_q satisfy the system of M nonlinear equations ($q=1,\ldots,M$)

$$\sum_{\alpha=1}^{4}\frac{1/4}{u_q-e_\alpha}+\sum_{j=1}^{K}\frac{1/2}{u_q-e_{\alpha_j}}+\sum_{p=1}^{M}{}'\frac{1}{u_q-u_p}=0. \tag{19}$$

When $K=0$, the first factor in (18) must be taken as 1 and the second sum in (19) should be omitted. Equations (19) are derived by direct substitution of $F(\bar{x})$ (18) into equation (15). To prove the completeness we need the following:

Theorem 2. (Stieltjes [4]). Let M, K and $\{\alpha_j\}$ be fixed. Then there exist exactly C^2_{M+2} eigenfunctions $F(\bar{x}, \{u_q\})$ with different sets of u_q which are the different solutions of equations (19).

Corollary 1. Separation (partial) functions $f_i(\rho_i)$ have the form

$$f_i(\rho_i)=\left(\prod_{j=1}^{K}(\rho_i-e_{\alpha_j})\right)^{1/2}\prod_{p=1}^{M}(\rho_i-u_p). \tag{20}$$

The zeros u_p of the Stieltjes polynomials (20) are the critical points of the function $|G|$, where

$$G(u_1,\ldots,u_M)=\prod_{p=1}^{M}\prod_{\alpha=1}^{4}(u_p-e_\alpha)^{k_\alpha}\prod_{r=p+1}^{M}(u_p-u_r). \tag{21}$$

Here we imply that $k_\alpha=\frac{1}{2}$ if $\alpha\notin\{\alpha_j\}$, and $k_\alpha=\frac{3}{2}$ otherwise. A unique $f_i(\rho_i)$ corresponds to each of the C^2_{M+2} ways of distributing its M zeros u_p among the three intervals $(e_\alpha, e_{\alpha+1})$.

Corollary 2. Each solution $F(\bar{x})$ with $\{\alpha_j\}_{j=1}^{K}$ and $\{u\}_{p=1}^{M}$ fixed determines the following eigenvalues λ and μ:

$$\begin{aligned}
\lambda&=s\left(\frac{1}{2}\sum_\alpha e_\alpha+\sum_j e_{\alpha_j}+2\sum_p u_p-(s+1)\sum_\alpha\frac{e_\alpha^4}{\Pi'_\beta(e_\alpha-e_\beta)}\right)\\
\mu&=\left(\prod_\alpha e_\alpha\right)\left(\sum_p 1/u_p+\frac{1}{2}\sum_j 1/e_{\alpha_j}\right)\left(\sum_p 1/u_p+\frac{1}{2}\sum_j 1/e_{\alpha_j}+\frac{1}{2}\sum_\alpha 1/e_\alpha\right)\\
&\qquad-\sum_p 1/u_p^2-\frac{1}{2}\sum_j 1/e_{\alpha_j}^2.
\end{aligned} \tag{22}$$

Recall that this λ and μ are the spectrum of the operators X and Y (1), respectively, provided that (14) is true.

Thus we have constructed the basis by polynomials multiplied by square roots depending on the type of solution. This is the index K that defines the type of solution each consisting of the C_4^K eigenfunctions. It should be noted that various types can be described by action of the direct product of the two dihedral groups $D_2\times D_2$, which is the invariance discrete group of operators X and Y [2].

Gaudin [3] pointed out that the magnet he has studied is connected with the procedure of variable separation. He dealt with the su(2) magnet. We formulate here the exact correspondence between our problem and the hyperbolic 4-site *XXX* Gaudin magnet.

Recall some facts from the algebraic Bethe ansatz for the su(1, 1) *XXX* Gaudin magnet [5]. Let $\mathscr{L}(u)$ be a 2×2 matrix, depending on arbitrary complex parameter u,

$$\mathscr{L}(u)=\bar{L}(u)\bar{\sigma}=\begin{pmatrix}\mathscr{L}_s(u) & \mathscr{L}_-(u)\\ \mathscr{L}_+(u) & -\mathscr{L}_s(u)\end{pmatrix}\qquad \bar{\mathscr{L}}(u)=\sum_{\alpha=1}^{4}\frac{\bar{l}_\alpha}{u-e_\alpha}. \tag{23}$$

The generators l_α^i, $i=1,2,3$, satisfy the commutators

$$[l_\alpha^i, l_\beta^k] = -\mathrm{i}\,\delta_{\alpha\beta}\varepsilon_{ikm} l_\alpha^m \mathscr{G}_{mm} \qquad \mathscr{G} = \mathrm{diag}(-1, -1, 1). \tag{24}$$

We will consider the boson representation:

$$l_3 = \tfrac{1}{4}\{a^+, a\} \qquad l_- = l_1 - \mathrm{i}l_2 = \tfrac{1}{2}a^2 \qquad l_+ = l_1 + \mathrm{i}l_2 = \tfrac{1}{2}a^{+2} \tag{25}$$

where $[a, a^+] = 1$ and the Casimir operator is $C_2 = l_3^2 - l_1^2 - l_2^2 = -\frac{3}{16}$.

$\mathscr{L}$-operator (23) obeys the XXX r-matrix algebra

$$[\mathscr{L}_i(u), \mathscr{L}_k(v)] = \frac{\mathrm{i}\varepsilon_{ikm}\mathscr{G}_{mm}}{u-v}(\mathscr{L}_m(u) - \mathscr{L}_m(v)). \tag{26}$$

The integrals of motion for this system are extracted from the $\bar{\mathscr{L}}^2(u)$ (in g-metric), obeying

$$[\bar{\mathscr{L}}^2(u), \bar{\mathscr{L}}^2(v)] = 0. \tag{27}$$

The eigenfunctions of the complete set of integrals H_α and $\bar{J}$

$$H_\alpha = \sum_{\beta\neq\alpha} \frac{\bar{l}_\alpha \bar{l}_\beta}{e_\alpha - e_\beta} \qquad \sum_{\alpha=1}^{4} H_\alpha = 0$$
$$\bar{J} = \sum_{\alpha=1}^{4} \bar{l}_\alpha \qquad \sum_{\alpha=1}^{4} e_\alpha H_\alpha = \tfrac{1}{2}(\bar{J}^2 + \tfrac{3}{4}) \tag{28}$$

can be constructed by means of the algebraic Bethe ansatz. Let us introduce the vacuum state

$$|\Omega\rangle = \prod_{\alpha=1}^{4} |\varkappa_\alpha\rangle_\alpha \tag{29}$$

where each of the $\varkappa_\alpha$ is equal to 0 or 1. This state possesses a number of important properties:

$$\mathscr{L}_-(u)|\Omega\rangle = 0$$
$$\mathscr{L}_3(u)|\Omega\rangle = a(u)|\Omega\rangle \qquad a(u) = \sum_{\alpha=1}^{4} \frac{1/4 + \varkappa_\alpha/2}{u - e_\alpha} \tag{30}$$
$$\bar{\mathscr{L}}^2(u)|\Omega\rangle = t(u)|\Omega\rangle \qquad t(u) = a^2(u) + a'(u).$$

Now we can formulate the following:

Theorem 3 [5]. Each vector

$$|u_1, \ldots, u_M\rangle = \mathscr{L}_+(u_1)\ldots\mathscr{L}_+(u_M)|\Omega\rangle \tag{31}$$

is an eigenvector of $\bar{\mathscr{L}}^2(u)$ with the eigenvalue

$$t(u) + \sum_{p=1}^{M}\left(\frac{2a(u)}{u-u_p} + \sum_{q\neq p}\frac{1}{(u-u_p)(u-u_q)}\right) \tag{32}$$

if and only if numbers $\{u_1, \ldots, u_M\}$ satisfy the equations

$$a(u_q) + \sum_{p\neq q}\frac{1}{u_q - u_p} = 0. \tag{33}$$

Equations (33) are just the same as ones (19) in theorem 1. Hence, we can say that we have established an isomorphism of the Hilbert spaces: of the Euler–Manakov top on the 3-sphere S_3 and of the hyperbolic Gaudin magnet. According to this isomorphism the infinite-dimensional representation space of the Gaudin system, the basis of which is given by the vectors (31), must be factorized by su(1, 1) total spin algebra $\bar{J}$ to obtain the $(2s+1)^2$-dimensional space $\mathscr{H}_s$, the basis of which is given by the vectors (18).

Our results demonstrate close connections between different problems and approaches. First, we have identified a construction of an ellipsoid basis on the S_3 sphere and spectral problem for the 2-site su(2) XYZ Gaudin magnet. The eigenfunctions are found with the help of nonlinear algebraic equations for their zeros. It allowed us to obtain the solution that generalizes the standard treatment of the Lamé polynomials. Further, the basic nonlinear equations (19) were interpreted in terms of the 4-site su(1, 1) XXX Gaudin magnet. Thus the unknown isomorphism between two different Gaudin magnets was established. This further evidence that different Lax matrices can correspond to the same dynamical system. The extension of our results to the S_n sphere will be published elsewhere.

Eigenfunctions of the form similar to (18) have been constructed in [6] for the most general complete set of commuting operators on the sphere in four-dimensional complex space.

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